Local Eigenvalue Asymptotics of the Perturbed Krein Laplacian

QMath13

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Based on the preprint:

V. Bruneau, G. Raikov,
Spectral properties of harmonic Toeplitz operators and applications to the perturbed Krein Laplacian, arXiv:1609.08229.

1. The Krein Laplacian and its perturbations

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary $\partial\Omega \in C^\infty$. For $s \in \mathbb{R}$, we denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$ the Sobolev spaces on Ω and $\partial\Omega$ respectively, and by $H_0^s(\Omega)$, $s > 1/2$, the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$.

Define the minimal Laplacian

$$\Delta_{\text{min}} := \Delta, \quad \text{Dom}\,\Delta_{\text{min}} = H_0^2(\Omega).$$

Then Δ_{min} is symmetric and closed but not self-adjoint in $L^2(\Omega)$ since

$$\Delta_{\text{max}} := \Delta_{\text{min}}^* = \Delta,$$

$$\text{Dom}\,\Delta_{\text{max}} = \left\{ u \in L^2(\Omega) \,|\, \Delta\, u \in L^2(\Omega) \right\}.$$

We have

$$\text{Ker}\,\Delta_{\text{max}} = \mathcal{H}(\Omega) := \left\{ u \in L^2(\Omega) \,|\, \Delta u = 0 \text{ in } \Omega \right\},$$

$$\text{Dom}\,\Delta_{\text{max}} = \mathcal{H}(\Omega) \dotplus H_D^2(\Omega)$$

where $H_D^2(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$.

Introduce the Krein Laplacian

$$K := -\Delta, \quad \operatorname{Dom} K = \mathcal{H}(\Omega) \dot{+} H_0^2(\Omega).$$

The operator $K \geq 0$, self-adjoint in $L^2(\Omega)$, is the von Neumann-Krein "soft" extension of $-\Delta_{\mathrm{min}}$, remarkable for its property that any other self-adjoint extension $S \geq 0$ of $-\Delta_{\mathrm{min}}$ satisfies

$$(S+I)^{-1} \leq (K+I)^{-1}.$$

We have $\operatorname{Ker} K = \mathcal{H}(\Omega)$. Moreover, $\operatorname{Dom} K$ can be described in terms of *the Dirichlet-to-Neumann operator* $\mathcal{D}$. For $f \in C^\infty(\partial\Omega)$, set

$$\mathcal{D} f = \frac{\partial u}{\partial \nu}_{|\partial\Omega},$$

where ν is the outer normal unit vector at $\partial\Omega$, u is the solution of the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases}$$

Thus, $\mathcal{D}$ is a first-order elliptic ΨDO; hence, it extends to a bounded operator form $H^s(\partial\Omega)$ into $H^{s-1}(\partial\Omega)$, $s \in \mathbb{R}$. In particular, $\mathcal{D}$ with domain $H^1(\partial\Omega)$ is self-adjoint in $L^2(\partial\Omega)$.

Then we have

$$\mathsf{Dom}\, K =$$

$$\left\{ u \in \mathsf{Dom}\, \Delta_{\mathsf{max}} \,\middle|\, \frac{\partial u}{\partial \nu}_{|\partial\Omega} = \mathcal{D}\left(u_{|\partial\Omega}\right) \right\}.$$

The Krein Laplacian K arises naturally in the so called *buckling problem*:

$$\begin{cases} \Delta^2 u = -\lambda \Delta u, \\ u_{|\partial\Omega} = \frac{\partial u}{\partial \nu}_{|\partial\Omega} = 0, \\ u \in \mathsf{Dom}\, \Delta_{\mathsf{max}}. \end{cases}$$

Let L be the restriction of K onto $\operatorname{Dom} K \cap \mathcal{H}(\Omega)^{\perp}$ where $\mathcal{H}(\Omega)^{\perp} := L^2(\Omega) \ominus \mathcal{H}(\Omega)$. Then, L is self-adjoint in $\mathcal{H}(\Omega)^{\perp}$.

Proposition 1. *The spectrum of L is purely discrete and positive, and, hence, L^{-1} is compact in $\mathcal{H}(\Omega)^{\perp}$. As a consequence, $\sigma_{\mathrm{ess}}(K) = \{0\}$, and the zero is an isolated eigenvalue of K of infinite multiplicity.*

Let $V \in C(\overline{\Omega}; \mathbb{R})$. Then the operator $K + V$ with domain $\operatorname{Dom} K$ is self-adjoint in $L^2(\Omega)$. In the sequel, we will investigate the spectral properties of $K + V$.

It should be underlined here that the perturbations $K+V$ are of different nature than the perturbations K_V discussed in the article M. S. Ashbaugh, F. Gesztesy, M. Mitrea, G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, Adv. Math. **223** (2010), 1372–1467, where the authors assume that $V \geq 0$, and set

$$K_{V,\max} := -\Delta + V, \ \mathrm{Dom}\, K_{V,\max} := \mathrm{Dom}\, \Delta_{\max},$$

$$K_V := -\Delta + V, \ \mathrm{Dom}\, K_V := \mathrm{Ker}\, K_{V,\max} \dot{+} H_0^2(\Omega).$$

Thus, if $V \neq 0$, then the operators K_V and $K_0 = K$ are self-adjoint on different domains, while the operators $K+V$ are all self-adjoint on $\mathrm{Dom}\, K$. Moreover, for any $0 \leq V \in C(\overline{\Omega})$, we have $K_V \geq 0$, $\sigma_{\mathrm{ess}}(K_V) = \{0\}$, and the zero is an isolated eigenvalue of K_V of infinite multiplicity. As we will see, the properties of $K+V$ could be quite different.

Theorem 1. *Let $V \in C(\overline{\Omega}; \mathbb{R})$. Then we have*

$$\sigma_{\text{ess}}(K+V) = V(\partial\Omega).$$

In particular, $\sigma_{\text{ess}}(K+V) = \{0\}$ if and only if $V_{|\partial\Omega} = 0$.

In the rest of the talk, we assume that $0 \leq V \in C(\overline{\Omega})$ with

$$V_{|\partial\Omega} = 0, \tag{1}$$

and will investigate the asymptotic distribution of the discrete spectrum of the operators $K \pm V$, adjoining the origin.

Set $\lambda_0 := \inf \sigma(L)$,

$$\mathcal{N}_-(\lambda) := \operatorname{Tr} \mathbb{1}_{(-\infty,-\lambda)}(K-V), \ \lambda > 0,$$

$$\mathcal{N}_+(\lambda) := \operatorname{Tr} \mathbb{1}_{(\lambda,\lambda_0)}(K+V), \ \lambda \in (0, \lambda_0).$$

Let $P : L^2(\Omega) \to L^2(\Omega)$ be the orthogonal projection onto $\mathcal{H}(\Omega)$. Introduce *the harmonic Toeplitz operator*

$$T_V := PV : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega).$$

If $V \in C(\overline{\Omega})$, then T_V is compact if and only if (1) holds true.

Let $T = T^*$ be a compact operator in a Hilbert space. Set

$$n(s;T) := \operatorname{Tr} \mathbb{1}_{(s,\infty)}(T), \quad s > 0.$$

Thus, $n(s;T)$ is just the number of the eigenvalues of the operator T larger than s, counted with their multiplicities.

Theorem 2. *Assume that* $0 \leq V \in C(\overline{\Omega})$ *and* $V_{|\partial\Omega} = 0$. *Then for any* $\varepsilon \in (0,1)$ *we have*

$$n(\lambda; T_V) \leq \mathcal{N}_-(\lambda) \leq n((1-\varepsilon)\lambda; T_V) + O(1),$$

and

$$n((1+\varepsilon)\lambda; T_V) + O(1) \leq$$

$$\mathcal{N}_+(\lambda) \leq$$

$$n((1-\varepsilon)\lambda; T_V) + O(1),$$

as $\lambda \downarrow 0$.

The proof of Theorem 2 is based on suitable versions of *the Birman–Schwinger principle*.

2. Spectral asymptotics of T_V for V of power-like decay at $\partial\Omega$

Let $a, \tau \in C^\infty(\bar{\Omega})$ satisfy $a > 0$ on $\bar{\Omega}$, $\tau > 0$ on Ω, and $\tau(x) = \mathrm{dist}(x, \partial\Omega)$ for x in a neighborhood of $\partial\Omega$. Assume

$$V(x) = \tau(x)^\gamma a(x), \quad \gamma \geq 0, \quad x \in \Omega. \tag{2}$$

Set $a_0 := a_{|\partial\Omega}$.

Theorem 3. *Assume that* V *satisfies* (2) *with* $\gamma > 0$. *Then we have*

$$n(\lambda; T_V) = \mathcal{C}\, \lambda^{-\frac{d-1}{\gamma}} \left(1 + O(\lambda^{1/\gamma})\right), \quad \lambda \downarrow 0, \tag{3}$$

where

$$\mathcal{C} := \omega_{d-1} \left(\frac{\Gamma(\gamma+1)^{1/\gamma}}{4\pi}\right)^{d-1} \int_{\partial\Omega} a_0(y)^{\frac{d-1}{\gamma}}\, dS(y), \tag{4}$$

and $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$ *is the volume of the unit ball in* $\mathbb{R}^n$, $n \geq 1$.

Idea of the proof of Theorem 3:

Assume that $f \in L^2(\partial\Omega)$, $s \in \mathbb{R}$. Then the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega, \end{cases}$$

admits a unique solution $u \in H^{1/2}(\Omega)$, and the mapping $f \mapsto u$ defines an isomorphism between $L^2(\partial\Omega)$ and $H^{1/2}(\Omega)$. Set

$$u := Gf.$$

The operator $G : L^2(\partial\Omega) \to L^2(\Omega)$ is compact, and

$$\operatorname{Ker} G = \{0\}, \quad \overline{\operatorname{Ran} G} = \mathcal{H}(\Omega).$$

Set $J := G^*G$. Then the operator $J = J^* \geq 0$ is compact in $L^2(\partial\Omega)$, and $\operatorname{Ker} J = \{0\}$. Hence, the operator J^{-1} is well defined as an unbounded positive operator, self-adjoint in $L^2(\partial\Omega)$.

Let

$$G = U|G| = UJ^{1/2}$$

be the polar decomposition of the operator G, where $U : L^2(\partial\Omega) \to L^2(\Omega)$ is an isometric operator with $\operatorname{Ker} U = \{0\}$ and $\operatorname{Ran} U = \mathcal{H}(\Omega)$.

Proposition 2. *The orthogonal projection P onto $\mathcal{H}(\Omega)$ satisfies*

$$P = GJ^{-1}G^* = UU^*.$$

Assume that V satisfies (2) with $\gamma \geq 0$, and set $J_V := G^*VG$.

Proposition 3. *Let V satisfy* (2) *with $\gamma > 0$. Then the operator T_V is unitarily equivalent to the operator $J^{-1/2}J_VJ^{-1/2}$.*

Proof. We have

$$PVP = UJ^{-1/2}J_VJ^{-1/2}U^*,$$

and U maps unitarily $L^2(\partial\Omega)$ onto $\mathcal{H}(\Omega)$. □

Proposition 4. *Under the assumptions of Proposition 3 the operator* $J^{-1/2} J_V J^{-1/2}$ *is a* ΨDO *with principal symbol*

$$2^{-\gamma}\Gamma(\gamma+1)|\eta|^{-\gamma} a_0(y), \quad (y,\eta) \in T^*\partial\Omega.$$

The proof of Proposition 4 is based on the pseudo-differential calculus due to L. Boutet de Monvel.

Further, under the assumptions of Theorem 3, we have $\operatorname{Ker} J^{-1/2} J_V J^{-1/2} = \{0\}$. Define the operator

$$A := \left(J^{-1/2} J_V J^{-1/2}\right)^{-1/\gamma}.$$

Then A is a ΨDO with principal symbol

$$2\Gamma(\gamma+1)^{-1/\gamma}|\eta| a_0(y)^{-1/\gamma}, \quad (y,\eta) \in T^*\partial\Omega.$$

By Proposition 3 and the spectral theorem, we have

$$n(\lambda; T_V) = \operatorname{Tr} \mathbb{1}_{\left(-\infty, \lambda^{-1/\gamma}\right)}(A), \quad \lambda > 0. \tag{5}$$

A classical result from L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218, implies that

$$\operatorname{Tr} \mathbb{1}_{(-\infty, E)}(A) = \mathcal{C} E^{d-1}(1+O(E^{-1})), \ E \to \infty, \tag{6}$$

the constant $\mathcal{C}$ being defined in (4). Combining (5) and (6), we arrive at (3).

3. Spectral asymptotics of T_V for radially symmetric compactly supported V

In this section we discuss the eigenvalue asymptotics of T_V in the case where Ω is the unit ball in $\mathbb{R}^d$, $d \geq 2$, while V is compactly supported in Ω, and possesses a partial radial symmetry.

Set

$$B_r := \left\{ x \in \mathbb{R}^d \,|\, |x| < r \right\}, \quad d \geq 2, \quad r \in (0, \infty).$$

Proposition 5. *Let* $\Omega = B_1$. *Assume that* $0 \leq V \in C(\overline{B_1})$, *and* $\operatorname{supp} V = \overline{B_c}$ *for some* $c \in (0,1)$. *Suppose moreover that for any* $\delta \in (0, c)$ *we have* $\inf_{x \in B_\delta} V(x) > 0$. *Then*

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-d+1} \, n(\lambda; T_V) = \frac{2^{-d+2}}{(d-1)! |\ln c|^{d-1}}.$$

The proof of Proposition 5 is based on the following

Lemma 1. *Let* $\Omega = B_1$, $V = b\mathbb{1}_{B_c}$ *with some* $b > 0$ *and* $c \in (0,1)$. *Then we have*

$$n(\lambda; T_V) = M_{\kappa(\lambda)}, \quad \lambda > 0,$$

where

$$M_k := \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1}, \quad k \in \mathbb{Z}_+,$$

with

$$\binom{m}{n} = \begin{cases} \frac{m!}{(m-n)!\,n!} & \textit{if} \quad m \geq n, \\ 0 \quad \textit{if} \quad m < n, \end{cases}$$

and

$$\kappa(\lambda) := \#\left\{k \in \mathbb{Z}_+ \,|\, bc^{2k+d} > \lambda\right\}, \quad \lambda > 0.$$

Thank you!